

# DECOMPOSITION MORPHISMS ARE GENERICALLY TRIVIAL

ULRICH THIEL

## Abstract

Let  $R$  be a normal noetherian ring with quotient field  $K$  and let  $A$  be an  $R$ -algebra which is free and finitely generated as an  $R$ -module, and which splits over  $K$ . We are interested in the specializations  $A(\mathfrak{p}) := k(\mathfrak{p}) \otimes_R A$  of  $A$  in prime ideals  $\mathfrak{p}$  of  $R$ . Here,  $k(\mathfrak{p})$  is the residue field of  $R$  in  $\mathfrak{p}$ . We show that the set of all prime ideals  $\mathfrak{p}$  of  $R$  such that the decomposition morphism  $d_A^{\mathfrak{p}} : G_0(A^K) \rightarrow G_0(A(\mathfrak{p}))$  as introduced by Geck–Rouquier exists and is trivial, forms a dense subset of  $\text{Spec}(R)$ . Hence, decomposition morphisms are generically trivial. Moreover, if  $A(\mathfrak{p})$  splits for all  $\mathfrak{p}$ , we show that this set is even open.

## §1. Introduction

In modular representation theory of finite groups the technique of *p-modular reduction* is well-established and an important tool. Among others, it leads to the notion of *decomposition matrices* and the *Brauer–Cartan triangle* (see for example [4]). The constructions rely on some ring-theoretic methods but all that is essentially needed are localizations of Dedekind domains and properties of modules over discrete valuation rings. This setting has been extended in various ways to include further classes of algebras, for example Hecke algebras. We mention here Geck–Rouquier [9] (who introduce a formal characterization of decomposition morphisms, see also Geck–Pfeiffer [8] for a general exposition) and Du–Parshall–Scott [5] (who work with regular rings of dimension at most two). The extension from one-dimensional base rings in the classical setting to general base rings (and general algebras) also brings along geometric questions, and one of these is the topic of this article.

To understand the reason for these generalizations note that we can consider the prime ideals of the base ring  $R$  of an  $R$ -algebra  $A$  as *parameters* for  $A$ . We can then *specialize* the algebra  $A$  in a prime ideal  $\mathfrak{p}$  by passing to the algebra  $A(\mathfrak{p}) = k(\mathfrak{p}) \otimes_R A$ , where  $k(\mathfrak{p})$  is the residue field in  $\mathfrak{p}$ , and can ask how the representation theory of  $A(\mathfrak{p})$  varies with  $\mathfrak{p}$ . This is really the generalization of what is going on in modular representation theory of a finite group  $G$ . Here, one studies the algebra  $A = \mathcal{O}G$  for the ring of integers  $\mathcal{O}$  in a number field  $K$ . The specialization in a prime  $\mathfrak{p}$  of  $\mathcal{O}$  is the

---

Date: February 24, 2014 (first version: February 20, 2014)

ULRICH THIEL, Universität Stuttgart, Fachbereich Mathematik, Institut für Algebra und Zahlentheorie, Lehrstuhl für Algebra, Pfaffenwaldring 57, 70569 Stuttgart, Germany.

Email: thiel@mathematik.uni-stuttgart.de

group algebra of  $G$  over a finite field of characteristic  $p$ , where  $(p) = \mathbb{Z} \cap \mathfrak{p}$ , so the question here is to understand the representation theory of  $G$  in all characteristics. In this case it is a classical fact that the representation theory of  $G$  is essentially the same as the one of  $KG$  for almost all  $p$ . But what about the analog of this question in the general setting? Is it true that the representation theory of  $A(\mathfrak{p})$  remains essentially the same for almost all  $\mathfrak{p}$ ? In this article we give a positive answer to this question under very mild conditions (see 2.7 for the precise formulation of this question and theorem 6.4 for the main result). The theory of decomposition morphisms and our result provide a way of systematically studying the representation theory of  $A(\mathfrak{p})$  for all  $\mathfrak{p}$ .

More recent applications of decomposition morphisms in the case of (restricted) rational Cherednik algebras (see [1] and [16]) led us to consider a very general setting in which we assume almost nothing instead of the base ring being normal and noetherian. As Hecke algebras are generically semisimple one can use Schur elements to derive some properties of decomposition morphisms (see [8]). This is not true for Cherednik algebras, however, and this is why we cannot use such methods here and why we consider such a general setup.

Our approach relies on a result by Grothendieck about the existence of discrete valuation rings. This has a major impact on the theory of decomposition morphisms and allows us to generalize arguments by Geck [7]. These form one ingredient of our result, the other consisting of the concept of generic properties we will introduce.

**Acknowledgements.** I am very thankful to Meinolf Geck for several comments on a preliminary version of this article. Moreover, I would like to thank Gunter Malle for pointing out some typographical errors. Part of this work was supported by the DFG *Schwerpunktprogramm Darstellungstheorie 1388*.

## §2. Decomposition morphisms

**2.1.** To simplify notations, we say that an algebra  $A$  over a commutative ring  $R$  is *free* if it is a free  $R$ -module, and we say that  $A$  is *finite* if  $A$  is a finitely generated  $R$ -module. Except for some intermediate results, we will always consider the following setup:

$R$  is a normal (not necessarily noetherian) ring with quotient field  $K$  and  $A$  is a finite free  $R$ -algebra which splits over  $K$ .

We prefer to point out where exactly noetherianness of the base ring is assumed and we therefore did not include it in our assumption.

**2.2.** For a prime ideal  $\mathfrak{p}$  of  $R$  we denote by  $k(\mathfrak{p})$  the residue field of  $R$  in  $\mathfrak{p}$ , i.e., the quotient field of  $R/\mathfrak{p}$ , and we call

$$A(\mathfrak{p}) := k(\mathfrak{p}) \otimes_R A \cong k(\mathfrak{p}) \otimes_{R/\mathfrak{p}} A/\mathfrak{p}A \cong k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} A_{\mathfrak{p}}$$

the *specialization* (or *fiber*) of  $A$  in  $\mathfrak{p}$ . This is a  $k(\mathfrak{p})$ -algebra of the same dimension as  $A$ . In more geometric terms,  $A(\mathfrak{p})$  is the stalk at  $\mathfrak{p}$  of the  $\mathcal{O}_X$ -algebra on the affine scheme  $X := \text{Spec}(R)$  defined by  $A$ . We call the specialization  $A(0) = A^K$  in the generic point  $(0)$  of  $\text{Spec}(R)$  the *generic fiber* of  $A$ . We denote by  $\theta_A^{\mathfrak{p}} : A \rightarrow A(\mathfrak{p})$  the

canonical morphism. Moreover, we define  $A|_{\mathfrak{p}} := R/\mathfrak{p} \otimes_R A \cong A/\mathfrak{p}A$ , which can be considered as the restriction of  $A$  to the closed subscheme  $V(\mathfrak{p})$  of  $\text{Spec}(R)$ .

**2.3 Definition.** If  $\mathfrak{p}$  is a prime ideal of  $R$ , let us call an  $A$ -gate in  $\mathfrak{p}$  any valuation ring  $\mathcal{O}$  between  $R$  and  $K$  with maximal ideal  $\mathfrak{m}$  such that  $R \cap \mathfrak{m} = \mathfrak{p}$  and such that the canonical morphism  $\gamma_A^{\mathfrak{p}, \mathfrak{m}} : G_0(A(\mathfrak{p})) \rightarrow G_0(A^{\mathcal{O}}(\mathfrak{m}))$  of Grothendieck groups (zeroth  $K$ -group of the category of finitely generated modules) induced by the residue field extension  $k(\mathfrak{p}) \hookrightarrow k(\mathfrak{m})$  is an isomorphism.

**2.4.** The first condition on  $\mathcal{O}$  is independent of  $A$  and it is a standard commutative algebra fact that there always exists such a valuation ring (see for example [10, 5.1]). This follows essentially from the fact that valuation rings are precisely maximal elements in the set of local subrings of a field (see [2, VI.1.2]). The second condition on the invertibility of  $\gamma_A^{\mathfrak{p}, \mathfrak{m}}$  holds for example if  $A(\mathfrak{p})$  splits. Although this is the usual assumption, note, however, that if  $R$  is a Prüfer domain (which in the noetherian case is precisely a Dedekind domain), then the localization  $R_{\mathfrak{p}}$  is already an  $A$ -gate in  $\mathfrak{p}$  without having to assume that  $A(\mathfrak{p})$  splits.

If  $\mathcal{O}$  is an  $A$ -gate in  $\mathfrak{p}$ , then it is a further standard fact (see [8, 7.3.7]) that any finitely generated  $A^K$ -module  $V$  admits an  $\mathcal{O}$ -free  $A^{\mathcal{O}}$ -structure, i.e., there exists an  $\mathcal{O}$ -free  $A^{\mathcal{O}}$ -module  $\tilde{V}$  such that  $V \cong A^K \otimes_{A^{\mathcal{O}}} \tilde{V}$  as  $A^K$ -modules. This follows essentially from the fact that finitely generated torsion free modules over a valuation ring are already free (see [10, 5.2]).

**2.5.** The main result by Geck–Rouquier [9] (see also Geck–Pfeiffer [8, 7.4.3]) on decomposition morphisms can be reformulated as stating that if there exists an  $A$ -gate in  $\mathfrak{p}$ , then there exists a unique group morphism

$$d_A^{\mathfrak{p}} : G_0(A^K) \rightarrow G_0(A(\mathfrak{p}))$$

of Grothendieck groups satisfying the following property: for any  $A$ -gate  $\mathcal{O}$  in  $\mathfrak{p}$  with maximal ideal  $\mathfrak{m}$  and for any finitely generated  $A^K$ -module  $V$  and any  $\mathcal{O}$ -free  $A^{\mathcal{O}}$ -structure  $\tilde{V}$  of  $V$  we have

$$d_A^{\mathfrak{p}}([V]) = (\gamma_A^{\mathfrak{p}, \mathfrak{m}})^{-1}([\tilde{V}/\mathfrak{m}\tilde{V}]) .$$

The essence of this theorem is that  $d_A^{\mathfrak{p}}$  is independent of the choice of the  $A$ -gate and that it is a well-defined group morphism. We call this morphism the *A-decomposition morphism* in  $\mathfrak{p}$ .

**2.6 Remark.** The precise assumptions for the existence of decomposition morphisms have been worked out in general in [16, §8] and we mention some of the results here. First of all, the morphism  $\gamma_A^{\mathfrak{p}, \mathfrak{m}}$  is always injective (see [16, §4]), so it is enough to look for surjectivity of  $\gamma_A^{\mathfrak{p}, \mathfrak{m}}$ . Moreover, it is proven in [16, §8] that the Brauer–Nesbitt map (see [8, 7.3.1]) is always injective so that an implicit assumption in [9] and [8] for the existence of decomposition morphisms is actually superfluous and 2.5 really holds in this generality. Another important aspect is that one can consider  $d_A^{\mathfrak{p}}$  for a fixed  $A$ -gate  $\mathcal{O}$  and can ask whether this assignment  $d_A^{\mathfrak{p}, \mathcal{O}}$  exists. This has been proven in [16, §8] to be true without any additional assumptions. The only question is then the independence of  $d_A^{\mathfrak{p}, \mathcal{O}}$  from  $\mathcal{O}$  and this holds for example when  $R$  is normal what we assume throughout. At last, we point out that in case  $\mathcal{O}$  is a *discrete* valuation ring,

the proof of the existence of  $d_A^{\mathfrak{p}, \mathcal{O}}$  can be simplified by employing an argument by Serre (see the proof of [4, 16.16]) which does not use Brauer–Nesbitt maps. In case  $R$  is noetherian one can indeed always find a discrete valuation ring  $\mathcal{O}$  above  $\mathfrak{p}$  due to a theorem by Grothendieck which plays a central role in this article (see 3.6). This fact seems not to be well-known.

**2.7.** We say that a decomposition morphism  $d_A^{\mathfrak{p}}$  is *trivial* if it induces a bijection between the simple  $A^K$ -modules and the simple  $A(\mathfrak{p})$ -modules. If we choose as bases of the Grothendieck groups systems of representatives of the isomorphism classes of the simple modules, this means precisely that the matrix  $D_A^{\mathfrak{p}}$  of  $d_A^{\mathfrak{p}}$  in these bases is a permutation matrix. In this article we are interested in geometric properties of the subset

$$\text{DecGen}(A) := \{\mathfrak{p} \in \text{Spec}(R) \mid d_A^{\mathfrak{p}} \text{ exists and is trivial}\} \subseteq \text{Spec}(R)$$

and we show that it is always dense in  $\text{Spec}(R)$  if  $R$  is noetherian (see theorem 6.4). This means intuitively that Zariski almost all specializations of  $A$  look like the generic fiber of  $A$  from the representation-theoretic point of view. Hence, once we know the simple modules of the generic fiber we know them for Zariski almost all specializations. In case  $A(\mathfrak{p})$  splits for all  $\mathfrak{p}$  we furthermore prove that this set is indeed an open subset so that its complement  $\text{DecEx}(A)$  is closed and one can continue the study of the specializations of  $A$  by restricting  $A$  to  $\text{DecEx}(A)$ . This result is of high theoretical and practical value for the representation theory of any algebra involving parameters like Hecke algebras [3] or Cherednik algebras [6].

**2.8.** So far, geometric properties of  $\text{DecGen}(A)$  have not been worked out in general. Geck [7, §2] shows that if  $R$  is a Dedekind domain, then  $d_A^{\mathfrak{p}}$  is trivial for all but finitely many  $\mathfrak{p} \in \text{Spec}(R)$ , implying that  $\text{DecGen}(A)$  is open. Even this one-dimensional situation is hard to solve and it gets much harder if we do not want to restrict the dimension of the base ring. Geck’s arguments, however, still form an important ingredient of our approach showing that  $\text{DecGen}(A)$  is dense but we first have to establish them in our general context. On top of this we introduce several new concepts to tackle the general situation and to prove that  $\text{DecGen}(A)$  is even open provided that  $A(\mathfrak{p})$  splits for all  $\mathfrak{p}$ .

### §3. Connection with the Jacobson radical

Our strategy is not to work with the set  $\text{DecGen}(A)$  directly but to use a connection to the behavior of the Jacobson radical under specialization. The starting point is the following proposition, where we denote by  $j(A)$  the Jacobson radical of a ring  $A$ .

**3.1 Proposition.** Let  $\mathfrak{p} \in \text{Spec}(R)$  and suppose that  $A(\mathfrak{p})$  splits. If  $d_A^{\mathfrak{p}}$  is trivial, then  $\dim_K j(A^K) = \dim_{k(\mathfrak{p})} j(A(\mathfrak{p}))$ .

*Proof.* Let  $(S_i)_{i \in I}$  be a system of representatives of the isomorphism classes of simple  $A^K$ -modules. Then by assumption  $(d_A^{\mathfrak{p}}([S_i]))_{i \in I}$  is a system of representatives of the isomorphism classes of simple  $A(\mathfrak{p})$ -modules. Note that  $d_A^{\mathfrak{p}}$  preserves dimensions by

definition. Since both  $A^K$  and  $A(\mathfrak{p})$  split, we have by [14, 7.8] the equality

$$\begin{aligned} \dim_K j(A^K) + \sum_{i \in I} (\dim_K S_i)^2 &= \dim_K A^K = \dim_{k(\mathfrak{p})} A(\mathfrak{p}) \\ &= \dim_{k(\mathfrak{p})} j(A(\mathfrak{p})) + \sum_{i \in I} (\dim_{k(\mathfrak{p})} d_A^{\mathfrak{p}}([S_i]))^2 \\ &= \dim_{k(\mathfrak{p})} j(A(\mathfrak{p})) + \sum_{i \in I} (\dim_K S_i)^2 \end{aligned}$$

and therefore  $\dim_K j(A^K) = \dim_{k(\mathfrak{p})} j(A(\mathfrak{p}))$  as claimed.  $\blacksquare$

**3.2.** The proposition implies that

$$\text{DecGen}(A) \cap \text{Spl}(A) \subseteq \text{JacDimGen}(A) \cap \text{Spl}(A), \quad (1)$$

where

$$\text{JacDimGen}(A) := \{\mathfrak{p} \in \text{Spec}(R) \mid \dim_K j(A^K) = \dim_{k(\mathfrak{p})} j(A(\mathfrak{p}))\}$$

and

$$\text{Spl}(A) := \{\mathfrak{p} \in \text{Spec}(R) \mid A(\mathfrak{p}) \text{ splits}\}.$$

If the converse inclusion of (1) would hold and we could show that  $\text{JacDimGen}(A) \cap \text{Spl}(A)$  is dense, then also  $\text{DecGen}(A)$  would be dense so that decomposition morphisms would be generically trivial. Our aim in this paragraph is to show that the converse inclusion indeed holds provided that  $R$  is noetherian. In the following paragraphs we then show the desired geometric properties.

**3.3.** To understand the set  $\text{JacDimGen}(A)$  we need a way to compare the Jacobson radical of the generic fiber with the Jacobson radical of a specialization. To this end, we need some results about the behavior of submodules under scalar extension. If  $\theta : R \rightarrow S$  is a morphism into a commutative ring  $S$ , we set  $A^S := \theta^* A = S \otimes_R A$ . The natural ring morphism  $\theta_A : A \rightarrow A^S$ ,  $a \mapsto 1 \otimes a$ , induces the scalar extension functor  $\theta_A^* : A\text{-Mod} \rightarrow A^S\text{-Mod}$ . If  $V$  is an  $A$ -module, then  $\theta_A^* V = A^S \otimes_A V$  is an  $A^S$ -module whose underlying  $S$ -module structure is the one of  $V^S$ . Furthermore, we have a natural map  $\theta_V : V \rightarrow V^S$ ,  $v \mapsto 1 \otimes v$ , and this map allows us to set up a relation between  $A$ -submodules of  $V$  and  $A^S$ -submodules of  $V^S$ . Namely, if  $U \leq V$ , we set  $\text{ext}_V^S(U) := \langle \theta_V(U) \rangle_{A^S} \leq V^S$  and  $\text{con}_V^S(W) := \theta_V^{-1}(W) \leq V$  for  $W \leq V^S$ . If  $\theta_V$  is injective, we can identify  $V$  with a subset of  $V^S$  and then we have  $\text{con}_V^S(W) = V \cap W$ . This holds for example if  $\theta$  is the localization morphism for a multiplicatively closed subset  $\Sigma \subseteq R$  and  $V$  is  $\Sigma$ -torsion free. In this case [2, II, §2.2, Proposition 4] implies moreover that  $\text{ext}_V^S \circ \text{con}_V^S(W) = W$  for any  $W \leq V^S$  and that the image of  $\text{con}_V^S$  consists of all submodules  $U \leq V$  such that the quotient  $V/U$  is  $\Sigma$ -torsion free.

Using right-exactness of  $\theta_A^*$  it is easy to see that if  $f : V \rightarrow W$  is a surjective morphism of  $A$ -modules, then  $\text{Im}(\theta_A^* f) = \theta_A^* W$  and  $\text{Ker}(\theta_A^* f) = \text{ext}_V^S(\text{Ker}(f))$ . In particular, if  $U$  is a submodule of  $V$ , then  $\theta_A^*(V/U) \cong \theta_A^* V / \text{ext}_V^S(U)$  canonically. We will use this several times later.

Now, if  $J$  is a  $K$ -subspace of  $A^K$ , then due to the canonical  $A$ -structure  $A$  of  $A^K$  we can construct for any  $\mathfrak{p} \in \text{Spec}(R)$  canonically a subvector space  $J(\mathfrak{p})$  of  $A(\mathfrak{p})$  derived from  $J$ , namely

$$J(\mathfrak{p}) := \text{ext}_{A_{\mathfrak{p}}}^{k(\mathfrak{p})} \circ \text{con}_{A_{\mathfrak{p}}}^K(J) = k(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} (A_{\mathfrak{p}} \cap J).$$

The following lemma sets up a first relation between  $j(A^K)$  and  $j(A(\mathfrak{p}))$ .

**3.4 Lemma.** For any  $\mathfrak{p} \in \text{Spec}(R)$  the relation  $j(A(\mathfrak{p})) \supseteq j(A^K)(\mathfrak{p})$  holds.

*Proof.* Since  $A^K$  is artinian, its Jacobson radical is nilpotent. Hence,  $\text{con}_{A_{\mathfrak{p}}}^K(j(A^K)) = A_{\mathfrak{p}} \cap j(A^K)$  and so  $j(A^K)(\mathfrak{p})$  is nilpotent. This implies that  $j(A^K)(\mathfrak{p})$  is contained in  $j(A(\mathfrak{p}))$ .  $\blacksquare$

Now, we are ready to make the first step in proving the converse inclusion of (1). The following theorem is a very general version of Tits's deformation theorem and is based on arguments by Geck [7] but is put in a much more general context.

**3.5 Theorem.** Let  $\mathfrak{p} \in \text{Spec}(R)$  and suppose that  $A(\mathfrak{p})$  splits. Suppose furthermore that there exists an  $A$ -gate  $\mathcal{O}$  in  $\mathfrak{p}$  which is a discrete valuation ring. If  $\dim_{k(\mathfrak{p})} j(A(\mathfrak{p})) = \dim_K j(A^K)$ , then  $d_A^{\mathfrak{p}}$  is trivial.

*Proof.* Let  $\mathfrak{m}$  be the maximal ideal of  $\mathcal{O}$ . We first show that the Jacobson radical of the specialization  $A^{\mathcal{O}}(\mathfrak{m})$  is equal to the specialization of  $j(A^K)$  in  $\mathfrak{m}$ . By 3.4 we have

$$\text{ext}_{A^{\mathcal{O}}}^{k(\mathfrak{m})} \circ \text{con}_{A^{\mathcal{O}}}^K j(A^K) \subseteq j(A^{\mathcal{O}}(\mathfrak{m})) . \quad (2)$$

Since  $A^{\mathcal{O}}$  is  $\mathcal{O}$ -torsion free, also  $\text{con}_{A^{\mathcal{O}}}^K(j(A^K)) \leq A^{\mathcal{O}}$  is  $\mathcal{O}$ -torsion free and thus free since  $\mathcal{O}$  is a discrete valuation ring. Moreover, by 3.3 we have

$$\text{ext}_{A^{\mathcal{O}}}^K \circ \text{con}_{A^{\mathcal{O}}}^K(j(A^K)) = j(A^K)$$

and therefore

$$\dim_{k(\mathfrak{m})} \text{ext}_{A^{\mathcal{O}}}^{k(\mathfrak{m})} \circ \text{con}_{A^{\mathcal{O}}}^K j(A^K) = \dim_{\mathcal{O}} \text{con}_{A^{\mathcal{O}}}^K j(A^K) = \dim_K j(A^K) . \quad (3)$$

Since  $A(\mathfrak{p})$  splits, it follows from [4, 7.9(i)] that

$$\text{ext}_{A(\mathfrak{p})}^{k(\mathfrak{m})} j(A(\mathfrak{p})) = j(A^{\mathcal{O}}(\mathfrak{m})) .$$

Combining this with equation (3) and our assumption, we see that

$$\dim_{k(\mathfrak{m})} j(A^{\mathcal{O}}(\mathfrak{m})) = \dim_{k(\mathfrak{p})} j(A(\mathfrak{p})) = \dim_K j(A^K) = \dim_{k(\mathfrak{m})} j(A^K)(\mathfrak{m}) .$$

Because of this, we must already have equality in (2), so

$$j(A^K)(\mathfrak{m}) = \text{ext}_{A^{\mathcal{O}}}^{k(\mathfrak{m})} \circ \text{con}_{A^{\mathcal{O}}}^K j(A^K) = j(A^{\mathcal{O}}(\mathfrak{m})) . \quad (4)$$

We know from 3.3 that the quotient  $\tilde{A} := A^{\mathcal{O}} / \text{con}_{A^{\mathcal{O}}}^K j(A^K)$  is  $\mathcal{O}$ -torsion free and thus already  $\mathcal{O}$ -free of finite dimension since  $\mathcal{O}$  is a discrete valuation ring. According to 3.3 we have

$$\tilde{A}^K = (A^{\mathcal{O}} / \text{con}_{A^{\mathcal{O}}}^K j(A^K))^K \cong A^K / (\text{ext}_{A^{\mathcal{O}}}^K \circ \text{con}_{A^{\mathcal{O}}}^K j(A^K)) = A^K / j(A^K) .$$

Hence,  $\tilde{A}^K$  is semisimple, and it is also split as a quotient of a split algebra. Furthermore, the canonical morphism

$$G_0(A^K) \rightarrow G_0(A^K / j(A^K)) = G_0(\tilde{A}^K)$$

clearly is trivial. Because of equation (3) also

$$\begin{aligned} \tilde{A}(\mathfrak{m}) &= \tilde{A}^{k(\mathfrak{m})} = (A^{\mathcal{O}} / \text{con}_{A^{\mathcal{O}}}^K j(A^K))^{k(\mathfrak{m})} \cong A^{\mathcal{O}}(\mathfrak{m}) / (\text{ext}_{A^{\mathcal{O}}}^{k(\mathfrak{m})} \circ \text{con}_{A^{\mathcal{O}}}^K j(A^K)) \\ &= A^{\mathcal{O}}(\mathfrak{m}) / j(A^{\mathcal{O}}(\mathfrak{m})) \end{aligned}$$

is split semisimple and again the canonical morphism

$$G_0(A^\mathcal{O}(\mathfrak{m})) \rightarrow G_0(A^\mathcal{O}(\mathfrak{m})/\mathfrak{j}(A^\mathcal{O}(\mathfrak{m}))) = G_0(\tilde{A}(\mathfrak{m}))$$

is trivial. Moreover, since  $A(\mathfrak{p})$  splits, the canonical morphism

$$G_0(A(\mathfrak{p})) \rightarrow G_0(A(\mathfrak{p})^{k(\mathfrak{m})}) = G_0(A^\mathcal{O}(\mathfrak{m}))$$

is trivial and so we have a canonical morphism

$$G_0(A(\mathfrak{p})) \rightarrow G_0(\tilde{A}(\mathfrak{m}))$$

which is trivial.

Now,  $\mathcal{O}$  is an  $\tilde{A}$ -gate in  $\mathfrak{p}$  and so the decomposition morphism

$$d_A^{\mathfrak{p}} : G_0(\tilde{A}^K) \rightarrow G_0(\tilde{A}(\mathfrak{m}))$$

exists. According to Tits's deformation theorem [8, 7.4.6], this morphism is trivial since  $\tilde{A}(\mathfrak{m})$  is split semisimple. Once we know that the diagram

$$\begin{array}{ccc} G_0(A^K) & \xrightarrow{d_A^{\mathfrak{p}}} & G_0(A(\mathfrak{p})) \\ \cong \downarrow & & \downarrow \cong \\ G_0(\tilde{A}^K) & \xrightarrow{d_A^{\mathfrak{p}}} & G_0(\tilde{A}(\mathfrak{m})) \end{array}$$

commutes, where the vertical morphisms are the morphisms discussed above, we also know that  $d_A^{\mathfrak{p}}$  is trivial. It suffices to check commutativity on simple  $A^K$ -modules, so let  $S$  be a simple  $A^K$ -module. Let  $\tilde{S}$  be an  $\mathcal{O}$ -free  $A^\mathcal{O}$ -structure of  $S$ . Then  $d_A^{\mathfrak{p}}([S]) = (\gamma_A^{\mathfrak{p}, \mathfrak{m}})^{-1}([\tilde{S}/\mathfrak{m}\tilde{S}])$ , and the image of this element under the right vertical morphism is equal to  $[\tilde{S}/\mathfrak{m}\tilde{S}]$ . On the other hand, the image of  $[S]$  under the left vertical morphism is again  $[S]$  and as  $\tilde{S}$  is also an  $\mathcal{O}$ -free  $\tilde{A}$ -structure of  $S$ , we have  $d_A^{\mathfrak{p}}([S]) = [\tilde{S}/\mathfrak{m}\tilde{S}]$ . Hence, the diagram commutes and therefore  $d_A^{\mathfrak{p}}$  is trivial as claimed.  $\blacksquare$

**3.6.** An essential ingredient in 3.5 is the assumption that we can find an  $A$ -gate which is a *discrete* valuation ring. So far, we just know that there exists a valuation ring. This is a major problem in the theory of decomposition morphisms and this is why many considerations assume that the base ring is a Dedekind domain or assume other quite special settings. We, however, remind here of a theorem by Grothendieck [12, 7.1.7] (see also [11, 15.6]) which states that for any *noetherian* integral domain  $R$  with quotient field  $K$ , any *finitely generated* field extension  $L$  of  $K$ , and any non-zero prime ideal  $\mathfrak{p}$  of  $R$  there exists a *discrete* valuation ring  $\mathcal{O}$  between  $R$  and  $L$  with maximal ideal  $\mathfrak{m}$  and  $R \cap \mathcal{O} = \mathfrak{p}$ . Hence, this problem does not exist when assuming that  $R$  is noetherian and this detail does not seem to be widely known! Our discussion consequently implies the following theorem.

**3.7 Theorem.** If  $R$  is noetherian, then

$$\text{DecGen}(A) \cap \text{Spl}(A) = \text{JacDimGen}(A) \cap \text{Spl}(A) .$$

What we have to do now is to establish geometric properties of the sets  $\text{Spl}(A)$  and  $\text{JacDimGen}(A)$  with our actual aim being to show that they are not only dense

but even open in  $\text{Spec}(R)$ . Proving openness turned out to be an intricate problem for which we found a general technique to be discussed in the next paragraph.

## §4. Generic properties of algebras

From the definition of the sets  $\text{Spl}(A)$  and  $\text{JacDimGen}(A)$  we see that they always involve some property  $\mathcal{P}$  of algebras (or of specializations of  $A$ ) and then consist of all prime ideals  $\mathfrak{p}$  such that  $\mathcal{P}(A(\mathfrak{p}))$  holds. These properties satisfy a certain stability under localization and restriction which we can use recursively to prove that the above sets are open. This phenomenon led us to introduce an abstract notion of *generic properties* of algebras culminating in the general openness theorem 4.7.

**4.1.** First, we have to recall some basic topological notions from algebraic geometry which can be found for example in [13]. A topological space  $X$  is called *irreducible* if for any decomposition  $X = A \cup B$  with closed subsets  $A$  and  $B$  of  $X$ , already  $A = X$  or  $B = X$ . This is equivalent to the property that every non-empty open subset of  $X$  is already dense in  $X$ . If  $x, y \in X$  are two points in a topological space  $X$  such that  $y \in \overline{\{x\}}$ , then  $y$  is called a *specialization* of  $x$ , and  $x$  is called a *generization* of  $y$ . A subset  $E \subseteq X$  is called *stable under generization* (*stable under specialization*) if it contains all generizations (specializations) of its points. Clearly, closed subsets are stable under specialization, and hence open subsets are stable under generization. A *generic point* of  $X$  is a point  $x \in X$  such that  $X = \overline{\{x\}}$ . The space  $X$  is called *sober* if every irreducible closed subset of  $X$  has a unique generic point. By a *generic neighborhood* in an irreducible sober topological space we mean a neighborhood of the generic point of  $X$ . A topological space  $X$  is called *noetherian* if every descending chain of closed subspaces becomes stationary. Note that  $\text{Spec}(R)$  is an irreducible sober topological space with generic point  $(0)$  as we assumed that  $R$  is an integral domain. Furthermore,  $\text{Spec}(R)$  is noetherian if and only if  $R$  is noetherian.

**4.2 Definition.** Let  $\mathfrak{A}$  be a class of algebras over integral domains. Define

$$\tilde{\mathfrak{A}} := \{(A, \mathfrak{p}) \mid A \in \mathfrak{A}, \mathfrak{p} \in \text{Spec}(R), \text{ where } R \text{ is the base ring of } A\}.$$

Suppose that  $\mathfrak{A}$  is stable under localization and restriction, i.e., for any  $A \in \mathfrak{A}$  with base ring  $R$  and any  $\mathfrak{p} \in \text{Spec}(R)$  both  $A_{\mathfrak{p}}$  (as an  $R_{\mathfrak{p}}$ -algebra) and  $A|_{\mathfrak{p}}$  (as an  $(R/\mathfrak{p})$ -algebra) are contained in  $\mathfrak{A}$ . For a map  $\mathcal{P} : \tilde{\mathfrak{A}} \rightarrow \{0, 1\}$  and  $A \in \mathfrak{A}$  with base ring  $R$  we define

$$\mathcal{P}\text{Gen}(A) := \{\mathfrak{p} \in \text{Spec}(R) \mid \mathcal{P}(A, \mathfrak{p}) = 1\}.$$

We say that  $\mathcal{P}$  is a *generic property* of  $\mathfrak{A}$  if it satisfies the following conditions for any  $A \in \mathfrak{A}$  with base ring  $R$  and  $(0) \in \mathcal{P}\text{Gen}(A)$ :

- (a) The set  $\mathcal{P}\text{Gen}(A)$  is a generic neighborhood in  $\text{Spec}(R)$ .
- (b) If  $\mathfrak{p} \in \mathcal{P}\text{Gen}(A)$ , then:
  - (i)  $(0)_{\mathfrak{p}} \in \mathcal{P}\text{Gen}(A_{\mathfrak{p}})$ .
  - (ii)  $\mathfrak{p}_{\mathfrak{p}} \in \mathcal{P}\text{Gen}(A_{\mathfrak{p}})$ .
  - (iii) If  $\mathfrak{q} \in \text{Spec}(R)$  with  $\mathfrak{p} \supseteq \mathfrak{q}$  and  $\mathfrak{q}_{\mathfrak{p}} \in \mathcal{P}\text{Gen}(A_{\mathfrak{p}})$ , then  $\mathfrak{q} \in \mathcal{P}\text{Gen}(A)$ .
  - (iv)  $\mathfrak{p}/\mathfrak{p} \in \mathcal{P}\text{Gen}(A|_{\mathfrak{p}})$ .
  - (v) If  $\mathfrak{q} \in \text{Spec}(R)$  with  $\mathfrak{q} \supseteq \mathfrak{p}$  and  $\mathfrak{q}/\mathfrak{p} \in \mathcal{P}\text{Gen}(A|_{\mathfrak{p}})$ , then  $\mathfrak{q} \in \mathcal{P}\text{Gen}(A)$ .



Our aim is now to establish some geometric properties of the set  $\mathcal{P}\text{Gen}(A)$ , and of course we would like it to be open. This seems, however, to be very hard to see directly. What we will first show instead is that  $\mathcal{P}\text{Gen}(A)$  is at least always *almost open* in the following sense.

**4.3 Definition.** A subset  $E$  of a topological space  $X$  is *almost open* if it is stable under generization and if it is a union of locally closed subsets.

**4.4.** Clearly, arbitrary unions and finite intersections of almost open subsets of a topological space are again almost open. What is central to this article is that we can show that almost open subsets are already open provided that the space is sober and noetherian. To prove this we use a result by Grothendieck–Dieudonné [13, Chapitre 0, Corollaire 2.4.6] which states that in a noetherian topological space  $X$  a subset  $E$  of  $X$  is open if and only if for each irreducible closed subset  $Y$  of  $X$  the intersection  $E \cap Y$  is either empty or contains a non-empty open subset of  $Y$ .

**4.5 Theorem.** Let  $X$  be a sober noetherian topological space. Then any almost open subset of  $X$  is already open.

*Proof.* Let  $E$  be an almost open subset of  $X$ . Since  $E$  is a union of locally closed subsets, we can write  $E = \bigcup_{\lambda \in \Lambda} U_\lambda \cap Z_\lambda$  for open subsets  $U_\lambda$  and closed subsets  $Z_\lambda$  of  $X$ . We will now use the result from [13] mentioned above to show that  $E$  is open. So, let  $Y$  be an irreducible closed subset of  $X$  such that  $E \cap Y$  is non-empty. Since  $X$  is sober, the set  $Y$  has a generic point, i.e., there exists  $\xi \in Y$  such that  $Y = \overline{\{\xi\}}$ . Let  $y \in E \cap Y$ . Then  $y \in E$  and since  $E$  is stable under generization and  $\xi$  is a generization of  $y$ , also  $\xi \in E$ . Hence, there exists  $\lambda \in \Lambda$  such that  $\xi \in U_\lambda \cap Z_\lambda$ . But then  $\xi \in Z_\lambda$  and as  $Z_\lambda$  is closed, already  $Y = \overline{\{\xi\}} \subseteq Z_\lambda$ . This shows that

$$Y \cap U_\lambda = Y \cap Z_\lambda \cap U_\lambda \subseteq Y \cap E.$$

We conclude that  $Y \cap U_\lambda$  is a non-empty open subset of  $Y$  contained in  $Y \cap E$ . According to [13], this implies that  $E$  is already open. ■

We need the following basic lemma before we can prove that the sets  $\mathcal{P}\text{Gen}(A)$  are almost open.

**4.6 Lemma.** Suppose that  $R$  is a local integral domain. If  $U$  is a neighborhood in  $\text{Spec}(R)$  containing the maximal ideal of  $R$ , then already  $U = \text{Spec}(R)$ .

*Proof.* Since  $R$  is local, every prime ideal of  $R$  is contained in the unique maximal ideal  $\mathfrak{m}$  of  $R$  and is thus a generization of  $\mathfrak{m}$ . Since  $U$  is open and thus stable under generization, this implies that  $U = \text{Spec}(R)$ . ■

**4.7 Theorem.** Let  $\mathfrak{A}$  be a class of algebras over integral domains which is stable under localization and restriction, and let  $\mathcal{P}$  be a generic property of  $\mathfrak{A}$ . If  $A \in \mathfrak{A}$  with base ring  $R$  and  $(0) \in \mathcal{P}\text{Gen}(A)$ , then  $\mathcal{P}\text{Gen}(A)$  is an almost open generic neighborhood in  $\text{Spec}(R)$ . If  $R$  is noetherian, then  $\mathcal{P}\text{Gen}(A)$  is open.

*Proof.* To prove that  $\mathcal{P}\text{Gen}(A)$  is almost open we have to show that it is stable under generization and that it is a union of locally closed subsets.

We first show that  $\mathcal{P}\text{Gen}(A)$  is stable under generization. To this end, let  $\mathfrak{p} \in \mathcal{P}\text{Gen}(A)$ . By 4.2(b)(i) we have  $(0)_{\mathfrak{p}} \in \mathcal{P}\text{Gen}(A_{\mathfrak{p}})$  and therefore  $\mathcal{P}\text{Gen}(A_{\mathfrak{p}})$  is a generic neighborhood in  $\text{Spec}(R_{\mathfrak{p}})$  by 4.2(a). Furthermore, we have  $\mathfrak{p}_{\mathfrak{p}} \in \mathcal{P}\text{Gen}(A_{\mathfrak{p}})$  by 4.2(b)(ii) so that  $\mathcal{P}\text{Gen}(A_{\mathfrak{p}})$  is a neighborhood in  $\text{Spec}(R_{\mathfrak{p}})$  containing  $\mathfrak{p}_{\mathfrak{p}}$ . By 4.6 this implies that already  $\mathcal{P}\text{Gen}(A_{\mathfrak{p}}) = \text{Spec}(R_{\mathfrak{p}})$ . Now, suppose that  $\mathfrak{q} \in \text{Spec}(R)$  is a generization of  $\mathfrak{p}$ , i.e.,  $\mathfrak{p} \in \overline{\{\mathfrak{q}\}}$ . Then  $\mathfrak{p} \supseteq \mathfrak{q}$  and  $\mathfrak{q}_{\mathfrak{p}} \in \text{Spec}(R_{\mathfrak{p}}) = \mathcal{P}\text{Gen}(A_{\mathfrak{p}})$ . By 4.2(b)(iii), this implies  $\mathfrak{q} \in \mathcal{P}\text{Gen}(A)$ . Hence,  $\mathcal{P}\text{Gen}(A)$  is stable under generization.

Now, we prove that  $\mathcal{P}\text{Gen}(A)$  is a union of locally closed subsets. Let  $\mathfrak{p} \in \mathcal{P}\text{Gen}(A)$ . Note that  $Z_{\mathfrak{p}} := \text{Spec}(R/\mathfrak{p})$  is canonically homeomorphic to the closed subspace  $V(\mathfrak{p})$  in  $\text{Spec}(R)$ . According to 4.2(b)(iv) we have  $\mathfrak{p}/\mathfrak{p} \in \mathcal{P}\text{Gen}(A|_{\mathfrak{p}})$  and therefore  $\mathcal{P}\text{Gen}(A|_{\mathfrak{p}})$  is by 4.2(a) a generic neighborhood in  $Z_{\mathfrak{p}}$ . Let  $U_{\mathfrak{p}}$  be an open neighborhood of  $\mathfrak{p}/\mathfrak{p}$  contained in  $\mathcal{P}\text{Gen}(A|_{\mathfrak{p}})$ . An element of  $U_{\mathfrak{p}}$  can be written as  $\mathfrak{q}/\mathfrak{p}$  for some  $\mathfrak{q} \in \text{Spec}(R)$  such that  $\mathfrak{q} \supseteq \mathfrak{p}$ . Since  $U_{\mathfrak{p}} \subseteq \mathcal{P}\text{Gen}(A|_{\mathfrak{p}})$ , we have  $\mathfrak{q}/\mathfrak{p} \in \mathcal{P}\text{Gen}(A|_{\mathfrak{p}})$  and so  $\mathfrak{q} \in \mathcal{P}\text{Gen}(A)$  by 4.2(b)(v). Hence,  $U_{\mathfrak{p}}$ , considered as a subset of  $\text{Spec}(R)$ , is contained in  $\mathcal{P}\text{Gen}(A)$ . By definition of the subspace topology on  $Z_{\mathfrak{p}}$ , we can write  $U_{\mathfrak{p}} = Z_{\mathfrak{p}} \cap V_{\mathfrak{p}}$  with an open subset  $V_{\mathfrak{p}}$  of  $\text{Spec}(R)$ . Hence,

$$\mathcal{P}\text{Gen}(A) = \bigcup_{\mathfrak{p} \in \mathcal{P}\text{Gen}(A)} U_{\mathfrak{p}} = \bigcup_{\mathfrak{p} \in \mathcal{P}\text{Gen}(A)} Z_{\mathfrak{p}} \cap V_{\mathfrak{p}},$$

and this is a union of locally closed subsets. ■

## §5. The split locus

In this paragraph we show that the *split locus*  $\text{Spl}(A)$  is always an almost open subset of  $\text{Spec}(R)$  by showing that “being split” is a generic property in the sense of the last paragraph. The proof of this already contains some key arguments to be used in the proof of our main result.

We will use the following result due to Bonnafé–Rouquier which is proven in [1, proposition C.2.11] in the context of the behavior of blocks under specializations. We give it in a more general form here but prove it by the same arguments.

**5.1 Proposition.** Let  $\mathcal{F} \subseteq A^K$  be a finite subset. Then

$$\text{Gen}_A(\mathcal{F}) := \{\mathfrak{p} \in \text{Spec}(R) \mid \mathcal{F} \subseteq A_{\mathfrak{p}}\}$$

is an open generic neighborhood in  $\text{Spec}(R)$ .

*Proof.* For an element  $\alpha \in K$  we define  $I_{\alpha} := \{r \in R \mid r\alpha \in R\}$ . This is an ideal in  $R$  and it has the property that  $\alpha \in R_{\mathfrak{p}}$  if and only if  $I_{\alpha} \not\subseteq \mathfrak{p}$ . To see this, suppose that  $\alpha \in R_{\mathfrak{p}}$ . Then we can write  $\alpha = \frac{r}{x}$  for some  $x \in R \setminus \mathfrak{p}$ . Hence,  $x\alpha = r \in R$  and therefore  $x \in I_{\alpha}$ . Since  $x \notin \mathfrak{p}$ , it follows that  $I_{\alpha} \not\subseteq \mathfrak{p}$ . Conversely, if  $I_{\alpha} \not\subseteq \mathfrak{p}$ , then there exists  $x \in I_{\alpha}$  with  $x \notin \mathfrak{p}$ . By definition of  $I_{\alpha}$  we have  $x\alpha =: r \in R$  and since  $x \notin \mathfrak{p}$ , we can write  $\alpha = \frac{r}{x} \in R_{\mathfrak{p}}$ .

Now, let  $(a_1, \dots, a_n)$  be an  $R$ -basis of  $A$ . Then we can write every element  $f \in \mathcal{F}$  as  $f = \sum_{i=1}^n \alpha_{f,i} a_i$  with  $\alpha_{f,i} \in K$ . Let

$$I := \prod_{f \in \mathcal{F}, i \in [1, n]} I_{\alpha_{f,i}} \trianglelefteq R.$$

Then by the properties of the ideals  $I_\alpha$  we have the following logical equivalences:

$$\begin{aligned} (\mathcal{F} \subseteq A_{\mathfrak{p}}) &\iff (\alpha_{f,i} \in R_{\mathfrak{p}} \quad \forall f \in \mathcal{F}, i \in [1, n]) \\ &\iff (I_\alpha \not\subseteq \mathfrak{p} \quad \forall f \in \mathcal{F}, i \in [1, n]) \\ &\iff (I \not\subseteq \mathfrak{p}), \end{aligned}$$

the last equivalence following from the fact that  $\mathfrak{p}$  is prime. Hence,

$$\text{Spec}(R) \setminus \text{Gen}_A(\mathcal{F}) = V(I),$$

implying that  $\text{Gen}_A(\mathcal{F})$  is an open subset of  $\text{Spec}(R)$ .  $\blacksquare$

**5.2 Corollary.** If  $\mathcal{F} \subseteq A^K$  is a (not necessarily finite) subset, then the set

$$\{\mathfrak{p} \in \text{Spec}(R) \mid \mathcal{F} \cap A_{\mathfrak{p}} \neq \emptyset\}$$

is an open generic neighborhood of  $\text{Spec}(R)$ .

*Proof.* The given set is clearly equal to  $\bigcup_{f \in \mathcal{F}} \text{Gen}_A(\{f\})$  and as each set  $\text{Gen}_A(\{f\})$  is an open generic neighborhood by 5.1, so is the given set.  $\blacksquare$

Our proof of the density of  $\text{Spl}(A)$  is based on arguments by Geck [7]. The key idea is to consider the behavior of vector space morphisms from  $A^K$  into split semisimple  $K$ -algebras upon reduction modulo prime ideals of  $R$ . The following proposition shows that the set of prime ideals where such a morphism has “good reduction” is indeed open.

**5.3 Proposition.** Let  $\psi : A^K \rightarrow \prod_{t=1}^n \text{Mat}_{n_t}(K)$  be a surjective morphism of  $K$ -modules. When considering  $\prod_{t=1}^n \text{Mat}_{n_t}(R_{\mathfrak{p}})$  canonically as a subset of  $\prod_{t=1}^n \text{Mat}_{n_t}(K)$ , then the sets

$$\begin{aligned} \text{Gen}_A^{\subseteq}(\psi) &:= \{\mathfrak{p} \in \text{Spec}(R) \mid \psi(A_{\mathfrak{p}}) \subseteq \prod_{t=1}^n \text{Mat}_{n_t}(R_{\mathfrak{p}})\}, \\ \text{Gen}_A^{\supseteq}(\psi) &:= \{\mathfrak{p} \in \text{Spec}(R) \mid \psi(A_{\mathfrak{p}}) \supseteq \prod_{t=1}^n \text{Mat}_{n_t}(R_{\mathfrak{p}})\}, \end{aligned}$$

and

$$\text{Gen}_A(\psi) := \{\mathfrak{p} \in \text{Spec}(R) \mid \psi(A_{\mathfrak{p}}) = \prod_{t=1}^n \text{Mat}_{n_t}(R_{\mathfrak{p}})\}$$

are open generic neighborhoods in  $\text{Spec}(R)$ .

*Proof.* Let  $\mathcal{B} := (b_i)_{i=1}^m$  be a basis of  $A^K$  such that  $(b_i)_{i=1}^r$  is a basis of  $\text{Ker}(\psi)$  and  $(\psi(b_i))_{i=r+1}^m$  is an  $R$ -basis of  $\prod_{t=1}^n \text{Mat}_{n_t}(R) \subseteq \prod_{t=1}^n \text{Mat}_{n_t}(K)$ . This is possible since  $A^K / \text{Ker}(\psi) \cong \prod_{t=1}^n \text{Mat}_{n_t}(K)$  and so one can choose  $(b_i)_{i=r+1}^m$  to map for example in each component to the elementary matrices.

To prove the assertion for the first set, let  $\mathcal{A} := (a_i)_{i=1}^m$  be an  $R$ -basis of  $A$ . The  $K$ -linearity of  $\psi$  and the fact that  $(a_i)_{i=1}^m$  is also an  $R_{\mathfrak{p}}$ -basis of  $A_{\mathfrak{p}}$  for all  $\mathfrak{p} \in \text{Spec}(R)$  implies the equality

$$\text{Gen}_A^{\subseteq}(\psi) = \{\mathfrak{p} \in \text{Spec}(R) \mid \psi(a_i) \in \prod_{t=1}^n \text{Mat}_{n_t}(R_{\mathfrak{p}}) \quad \text{for all } i = 1, \dots, m\}.$$

We can write each basis element  $a_i$  uniquely as  $a_i = \sum_{j=1}^m \alpha_{ij} b_j$  with  $\alpha_{ij} \in K$ . As  $(\psi(b_i))_{i=r+1}^m$  is an  $R$ -basis of  $\prod_{t=1}^n \text{Mat}_{n_t}(R)$ , it is also an  $R_p$ -basis of  $\prod_{t=1}^n \text{Mat}_{n_t}(R_p)$  and since

$$\psi(a_i) = \psi\left(\sum_{j=1}^m \alpha_{ij} b_j\right) = \sum_{j=1}^m \alpha_{ij} \psi(b_j) = \sum_{j=r+1}^m \alpha_{ij} \psi(b_j),$$

the uniqueness of the basis representation implies that the element  $\psi(a_i)$  is contained in  $\prod_{t=1}^n \text{Mat}_{n_t}(R_p)$  if and only if  $\alpha_{ij} \in R_p$  for all  $i = 1, \dots, m$  and all  $j = r+1, \dots, m$ . Hence,

$$\text{Gen}_A^{\subseteq}(\psi) = \text{Gen}_R((\alpha_{ij})_{\substack{i=1, \dots, m \\ j=r+1, \dots, m}})$$

and this is an open generic neighborhood by 5.1.

Now we consider the assertion for the second set. It is obvious that

$$\begin{aligned} \text{Gen}_A^{\supseteq}(\psi) &= \bigcap_{x \in \prod_{t=1}^n \text{Mat}_{n_t}(R_p)} \{\mathfrak{p} \in \text{Spec}(R) \mid \psi^{-1}(x) \cap A_p \neq \emptyset\} \\ &\subseteq \bigcap_{i=r+1}^m \{\mathfrak{p} \in \text{Spec}(R) \mid \psi^{-1}(\psi(b_i)) \cap A_p \neq \emptyset\}. \end{aligned}$$

But this is actually an equality. To see this, suppose that  $\mathfrak{p}$  is contained in the last (finite) intersection. Then we can choose  $c_i \in \psi^{-1}(\psi(b_i)) \cap A_p$  for all  $r+1 \leq i \leq m$ . Let  $A'$  be the  $R_p$ -span of the  $c_i$  in  $A_p$ . Since  $\psi(c_i) = b_i$  and since  $(\psi(b_i))_{i=r+1}^m$  is an  $R$ -basis of  $\prod_{t=1}^n \text{Mat}_{n_t}(R)$  by assumption so that  $(\psi(b_i))_{i=r+1}^m$  is also an  $R_p$ -basis of  $\prod_{t=1}^n \text{Mat}_{n_t}(R_p)$ , it follows by  $R_p$ -linearity that  $\psi(A') = \prod_{t=1}^n \text{Mat}_{n_t}(R_p)$ . Hence,  $\psi(A_p) \supseteq \prod_{t=1}^n \text{Mat}_{n_t}(R_p)$  and therefore

$$\text{Gen}_A^{\supseteq}(\psi) = \bigcap_{i=r+1}^m \{\mathfrak{p} \in \text{Spec}(R) \mid \psi^{-1}(\psi(b_i)) \cap A_p \neq \emptyset\}.$$

As each of the sets in the above finite intersection is an open generic neighborhood by 5.2, also  $\text{Gen}_A^{\supseteq}(\psi)$  is an open generic neighborhood.

Finally, since  $\text{Gen}_A(\psi) = \text{Gen}_A^{\supseteq}(\psi) \cap \text{Gen}_A^{\subseteq}(\psi)$ , it follows that also  $\text{Gen}_A(\psi)$  is an open generic neighborhood.  $\blacksquare$

As a last ingredient we will need the following basic lemma.

**5.4 Lemma.** Suppose that  $A$  is a finite-dimensional algebra over a field  $K$ . Then  $A$  splits if and only if there exists a surjective  $K$ -algebra morphism  $\psi : A \rightarrow S$  into a split semisimple  $K$ -algebra  $S$  such that  $\text{Ker}(\psi)$  is nilpotent. The kernel of any such morphism is already equal to  $\mathfrak{j}(A)$  so that  $A/\mathfrak{j}(A) \cong S$ .

*Proof.* If  $A$  splits, then the morphism obtained by the composition  $A \rightarrow A/\mathfrak{j}(A)$  with the isomorphism  $A/\mathfrak{j}(A) \cong \prod_{i=1}^n \text{Mat}_{n_i}(K)$  by the Artin–Wedderburn theorem satisfies the claimed properties since  $A/\mathfrak{j}(A)$  splits by [15, 7.9]. Conversely, assume that  $\psi$  is such a morphism. Since  $\text{Ker}(\psi)$  is nilpotent, we have  $\text{Ker}(\psi) \subseteq \mathfrak{j}(A)$ . Since  $\psi$  is surjective, we have  $\psi(\mathfrak{j}(A)) \subseteq \mathfrak{j}(S) = 0$  and therefore  $\mathfrak{j}(A) \subseteq \text{Ker}(\psi)$ . Hence,  $A/\mathfrak{j}(A) = A/\text{Ker}(\psi) \cong S$  is split and now it follows from [14, 7.9] that  $A$  also splits.  $\blacksquare$

**5.5 Theorem.** The split locus

$$\text{Spl}(A) := \{\mathfrak{p} \in \text{Spec}(R) \mid A(\mathfrak{p}) \text{ splits}\}$$

is an almost open generic neighborhood in  $\text{Spec}(R)$ . If  $R$  is noetherian, it is open.

*Proof.* Let  $\mathfrak{A}$  be the class of all finite free algebras over integral domains. This class is certainly stable under localization and restriction. We will show that the function  $\mathcal{P} : \mathfrak{A} \rightarrow \{0, 1\}$  with  $\mathcal{P}(A, \mathfrak{p}) = 1$  if and only if  $A(\mathfrak{p})$  splits, is a generic property. To this end, we assume that the generic fiber  $A(0) = A^K$  splits.

First, we show that  $\text{Spl}(A)$  is a generic neighborhood in  $\text{Spec}(R)$ . We do this by showing that the set  $\bigcup_{\psi} \text{Gen}_A(\psi)$ , where  $\psi$  runs over all surjective  $K$ -algebra morphisms  $A^K \rightarrow S$  with nilpotent kernel into semisimple  $K$ -algebras, is contained in  $\text{Spl}(A)$ . Since  $A^K$  splits, such morphisms exist by 5.4, and since each  $\text{Gen}_A(\psi)$  is a generic neighborhood by 5.3, this will show that  $\text{Spl}(A)$  is a generic neighborhood. So, let  $\psi$  be such a morphism and let  $\mathfrak{p} \in \text{Gen}_A(\psi)$ . This means by definition that  $\psi$  restricts to a surjective morphism  $\phi := \psi|_{A_{\mathfrak{p}}} : A_{\mathfrak{p}} \rightarrow \prod_{i=1}^n \text{Mat}_{n_i}(R_{\mathfrak{p}})$ . Since  $\text{Ker}(\psi) = j(A^K) \trianglelefteq A^K$  is nilpotent, also

$$\text{Ker}(\phi) = A_{\mathfrak{p}} \cap j(A^K) = \text{con}_{A_{\mathfrak{p}}}^K(j(A^K)) \trianglelefteq A_{\mathfrak{p}}$$

is nilpotent. The morphism  $\bar{\phi} : A(\mathfrak{p}) \rightarrow \prod_{i=1}^n \text{Mat}_{n_i}(k(\mathfrak{p}))$  induced by  $\phi$  by reducing modulo  $\mathfrak{p}_{\mathfrak{p}}$  is actually the morphism  $(\theta_{A_{\mathfrak{p}}}^{\mathfrak{p}_{\mathfrak{p}}})^* \phi$ . It is clearly surjective and using 3.3 we see that

$$\text{Ker}(\bar{\phi}) = \text{Ker}((\theta_{A_{\mathfrak{p}}}^{\mathfrak{p}_{\mathfrak{p}}})^* \phi) = \text{ext}_{A_{\mathfrak{p}}}^{\theta_{A_{\mathfrak{p}}}^{\mathfrak{p}_{\mathfrak{p}}}}(\text{Ker}(\phi)) = \text{ext}_{A_{\mathfrak{p}}}^{k(\mathfrak{p})} \circ \text{con}_{A_{\mathfrak{p}}}^K(j(A^K)) = j(A^K)(\mathfrak{p}),$$

i.e.,  $\text{Ker}(\bar{\phi})$  is just the image of  $\text{Ker}(\phi)$  in  $A(\mathfrak{p})$ . As  $\text{Ker}(\phi)$  is nilpotent, we thus conclude that also  $\text{Ker}(\bar{\phi})$  is nilpotent. An application of 5.4 now shows that

$$j(A^K)(\mathfrak{p}) = j(A(\mathfrak{p})) \tag{5}$$

and that  $A(\mathfrak{p})$  splits. Hence,  $\mathfrak{p} \in \text{Spl}(A)$  and therefore  $\text{Gen}_A(\psi) \subseteq \text{Spl}(A)$ . This shows that 4.2(a) holds.

Now, let  $\mathfrak{p} \in \mathcal{P}\text{Gen}(A)$ . Then  $A_{\mathfrak{p}}((0)_{\mathfrak{p}}) \cong A^K$  splits by assumption so that  $(0)_{\mathfrak{p}} \in \mathcal{P}\text{Gen}(A_{\mathfrak{p}})$  and 4.2(i) holds. Moreover,  $A_{\mathfrak{p}}(\mathfrak{p}_{\mathfrak{p}}) \cong A(\mathfrak{p})$  splits by assumption so that  $\mathfrak{p}_{\mathfrak{p}} \in \mathcal{P}\text{Gen}(A_{\mathfrak{p}})$  and 4.2(ii) holds. If  $\mathfrak{q} \in \text{Spec}(R)$  with  $\mathfrak{p} \supseteq \mathfrak{q}$  and  $\mathfrak{q}_{\mathfrak{p}} \in \mathcal{P}\text{Gen}(A_{\mathfrak{p}})$ , then, since  $A_{\mathfrak{p}}(\mathfrak{q}_{\mathfrak{p}}) \cong A(\mathfrak{q})$ , also  $A(\mathfrak{q})$  splits so that  $\mathfrak{q} \in \mathcal{P}\text{Gen}(A)$  and 4.2(iii) holds. As  $A|_{\mathfrak{p}}(\mathfrak{p}/\mathfrak{p}) \cong A(\mathfrak{p})$ , also  $\mathfrak{p}/\mathfrak{p} \in \mathcal{P}\text{Gen}(A_{\mathfrak{p}})$  so that 4.2(iv) holds. Finally, if  $\mathfrak{q} \in \text{Spec}(R)$  with  $\mathfrak{q} \supseteq \mathfrak{p}$  and  $\mathfrak{q}/\mathfrak{p} \in \mathcal{P}\text{Gen}(A|_{\mathfrak{p}})$ , then, since  $A|_{\mathfrak{p}}(\mathfrak{q}/\mathfrak{p}) \cong A(\mathfrak{q})$ , also  $A(\mathfrak{q})$  splits and 4.2(v) holds. This finally shows that  $\mathcal{P}$  is a generic property and now the assertion follows from 4.7.  $\blacksquare$

**5.6 Definition.** We say that  $A$  has split fibers if  $\text{Spl}(A) = \text{Spec}(R)$ , i.e.,  $A(\mathfrak{p})$  splits for all  $\mathfrak{p} \in \text{Spec}(R)$ .

## §6. Generic behavior of the Jacobson radical

In this final paragraph we show that the set  $\text{JacDimGen}(A)$  is open, which according to 3.7 implies our main results about decomposition morphisms. Again we will show that this set is defined by a generic property.

**6.1.** We first need a better understanding about the relationship between two specializations  $A(\mathfrak{p})$  and  $A(\mathfrak{q})$  when  $\mathfrak{q} \supseteq \mathfrak{p}$ . We have the following commutative diagram

$$\begin{array}{ccccc}
 R/\mathfrak{q} & \xrightarrow{\quad} & k(\mathfrak{q}) & & \\
 \uparrow & \nwarrow & \nwarrow & \nwarrow & \\
 R/\mathfrak{p} & \xleftarrow{\quad} & R & \xrightarrow{\quad} & R_{\mathfrak{q}} \\
 & \searrow & \searrow & \downarrow & \\
 & & k(\mathfrak{p}) & \xleftarrow{\quad} & R_{\mathfrak{p}}
 \end{array}$$

of quotient and localization morphisms, where we already used some canonical isomorphisms. This diagram implies that there are three ways to pass from  $A$  to  $A(\mathfrak{p})$ , namely

$$\mathrm{ext}_{A_{\mathfrak{p}}}^{k(\mathfrak{p})} \circ \mathrm{ext}_{A_{\mathfrak{q}}}^{R_{\mathfrak{p}}} \circ \mathrm{ext}_A^{R_{\mathfrak{q}}} = \mathrm{ext}_{A|_{\mathfrak{p}}}^{k(\mathfrak{p})} \circ \mathrm{ext}_A^{R/\mathfrak{p}} = \mathrm{ext}_{A_{\mathfrak{p}}}^{k(\mathfrak{p})} \circ \mathrm{ext}_A^{R_{\mathfrak{p}}} ,$$

and that there are three ways to pass from  $A$  to  $A(\mathfrak{q})$ , namely

$$\mathrm{ext}_{A_{\mathfrak{q}}}^{k(\mathfrak{q})} \circ \mathrm{ext}_A^{R_{\mathfrak{q}}} = \mathrm{ext}_{A|_{\mathfrak{q}}}^{k(\mathfrak{q})} \circ \mathrm{ext}_{A|_{\mathfrak{p}}}^{R/\mathfrak{q}} \circ \mathrm{ext}_A^{R/\mathfrak{p}} = \mathrm{ext}_{A|_{\mathfrak{q}}}^{k(\mathfrak{q})} \circ \mathrm{ext}_A^{R/\mathfrak{q}} .$$

Note that  $A_{\mathfrak{q}}(\mathfrak{p}_{\mathfrak{q}}) \cong A(\mathfrak{p})$  and that  $A|_{\mathfrak{p}}(\mathfrak{q}/\mathfrak{p}) \cong A(\mathfrak{q})$  canonically.

To prove that  $\mathrm{JacDimGen}(A)$  is dense, we show that it contains the (dense) subset in the following proposition. Recall from 3.4 that  $j(A(\mathfrak{p})) \supseteq j(A^K)(\mathfrak{p})$ , so following the philosophy of this article it is natural to consider this set.

**6.2 Proposition.** The set

$$\mathrm{JacGen}(A) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid j(A(\mathfrak{p})) = j(A^K)(\mathfrak{p})\}$$

is an almost open generic neighborhood in  $\mathrm{Spec}(R)$ . If  $R$  is noetherian, it is open.

*Proof.* Let  $\mathfrak{A}$  be the class of all finite free algebras over integral domains. We will show that the function  $\mathcal{P} : \mathfrak{A} \rightarrow \{0, 1\}$  with  $\mathcal{P}(A, \mathfrak{p}) = 1$  if and only if  $j(A(\mathfrak{p})) = j(A^K)(\mathfrak{p})$  is a generic property. Note that we always have  $(0) \in \mathcal{P}\mathrm{Gen}(A)$ .

First, we show that  $\mathrm{JacGen}(A)$  is a generic neighborhood in  $\mathrm{Spec}(R)$ . Equation (5) in the proof of 5.5 show that if  $\mathfrak{p}$  is contained in the generic neighborhood  $\bigcup_{\psi} \mathrm{Gen}_A(\psi)$ , where  $\psi$  runs over all surjective  $K$ -algebra morphisms  $A^K \rightarrow S$  with nilpotent kernel into split semisimple  $K$ -algebras, then

$$j(A(\mathfrak{p})) = \mathrm{Ker}(\psi)(\mathfrak{p}) = j(A^K)(\mathfrak{p}) .$$

Hence, this generic neighborhood is contained in  $\mathrm{JacGen}(A)$ , which is thus itself a generic neighborhood.

To prove the properties in 4.2(b) we use the commutative diagram from 6.1 and the canonical isomorphisms throughout. So, let  $\mathfrak{p} \in \mathcal{P}\mathrm{Gen}(A)$ . As  $A_{\mathfrak{p}}((0)_{\mathfrak{p}}) = A^K$ , we have

$$j(A_{\mathfrak{p}}((0)_{\mathfrak{p}}))((0)_{\mathfrak{p}}) = j(A^K) = j(A_{\mathfrak{p}}((0)_{\mathfrak{p}}))$$

so that  $(0)_{\mathfrak{p}} \in \mathcal{P}\mathrm{Gen}(A_{\mathfrak{p}})$ , proving 4.2(b)(i). Moreover, since  $A_{\mathfrak{p}}(\mathfrak{p}_{\mathfrak{p}}) \cong A(\mathfrak{p})$ , we have

$$j(A_{\mathfrak{p}}(\mathfrak{p}_{\mathfrak{p}})) = j(A(\mathfrak{p})) = j(A^K)(\mathfrak{p}) = j(A_{\mathfrak{p}}((0)_{\mathfrak{p}}))(\mathfrak{p}_{\mathfrak{p}}) ,$$

proving 4.2(b)(ii). If  $q \in \text{Spec}(R)$  with  $p \supseteq q$  and  $q_p \in \mathcal{P}\text{Gen}(A_p)$ , then, since  $A_p(q_p) \cong A(q)$ , we have

$$j(A(q)) = j(A_p(q_p)) = j((A_p(0)_p)(q_p)) = j(A^K)(q)$$

so that  $q \in \mathcal{P}\text{Gen}(A)$  and 4.2(b)(iii) holds. Now, recall that  $A|_p(q/p) \cong A(q)$  for  $q \supseteq p$ . In particular, the generic fiber of  $A|_p$  is equal to  $A(p)$ . Hence,

$$j(A|_p(p/p)) = j(A(p)) = j(A|_p(p/p))(p/p)$$

so that 4.2(b)(iv) holds. Finally, if  $q \in \text{Spec}(R)$  with  $q \supseteq p$  and  $q/p \in \mathcal{P}\text{Gen}(A|_p)$ , then

$$\begin{aligned} j(A(q)) &= j(A|_p(q/p)) = j(A|_p(p/p))(q/p) = \text{ext}_{A|_p}^{k(q)} \circ \text{con}_{A|_p}^{k(p)} j(A(p)) \\ &= \text{ext}_{A|_p}^{k(q)} \circ \text{con}_{A|_p}^{k(p)} j(A^K)(p) = \text{ext}_{A|_p}^{k(q)} \circ \text{con}_{A|_p}^{k(p)} \circ \text{ext}_A^{k(p)} \circ \text{con}_A^K j(A^K) \\ &= \text{ext}_A^{k(q)} \circ \text{con}_A^K j(A^K) = j(A^K)(q). \end{aligned}$$

This shows that  $q \in \mathcal{P}\text{Gen}(A)$  and so 4.2(b)(v) holds. In total, we have proven that  $\mathcal{P}$  is a generic property and now the assertion follows from 4.7.  $\blacksquare$

**6.3 Theorem.** If  $R$  is noetherian, then the set

$$\text{JacDimGen}(A) := \{p \in \text{Spec}(R) \mid \dim_{k(p)} j(A(p)) = \dim_K j(A^K)\}$$

is an open generic neighborhood in  $\text{Spec}(R)$ .

*Proof.* Let  $\mathfrak{A}$  be the class of all finite free algebras over noetherian integral domains. We show that the map  $\mathcal{P} : \mathfrak{A} \rightarrow \{0, 1\}$  with  $\mathcal{P}(A, p) = 1$  if and only if  $\dim_{k(p)} j(A(p)) = \dim_K j(A^K)$  is a generic property. Since  $\text{JacDimGen}(A) = \mathcal{P}\text{Gen}(A)$ , this will prove the assertion by 4.7.

We always have  $(0) \in \mathcal{P}\text{Gen}(A)$  and we have to show that  $\mathcal{P}\text{Gen}(A)$  is a generic neighborhood in  $\text{Spec}(R)$ . We know from 5.5 and 6.2 that both  $\text{Spl}(A)$  and  $\text{JacGen}(A)$  are generic neighborhoods. Hence,  $\text{JacGen}(A) \cap \text{Spl}(A)$  is a generic neighborhood and it suffices to show that this intersection is contained in  $\text{JacDimGen}(A)$ . So, let  $p \in \text{Spl}(A) \cap \text{JacGen}(A)$ . If  $p = (0)$ , it is clearly contained in  $\text{JacDimGen}(A)$  and therefore we can assume that  $p \neq (0)$ . Because of 3.6 we can choose a discrete valuation ring  $\mathcal{O}$  with maximal ideal  $\mathfrak{m}$  between  $R$  and  $K$  such that  $R \cap \mathfrak{m} = p$ . Since  $\text{con}_{A_p}^{\mathcal{O}} \circ \text{con}_{A_p}^K j(A^K) = \text{con}_{A_p}^K j(A^K)$ , we have

$$\text{con}_{A_p}^K j(A^K) = \text{ext}_{A_p}^{\mathcal{O}} \circ \text{con}_{A_p}^{\mathcal{O}} \circ \text{con}_{A_p}^K j(A^K) = \text{ext}_{A_p}^{\mathcal{O}} \circ \text{con}_{A_p}^K j(A^K)$$

by 3.3. Since  $p \in \text{JacGen}(A)$ , we have

$$j(A^K)(p) = \text{ext}_{A_p}^{k(p)} \circ \text{con}_{A_p}^{k(p)} j(A^K) = j(A(p))$$

and we therefore get

$$\begin{aligned} \text{ext}_{A_p}^{k(m)} \circ \text{con}_{A_p}^K j(A^K) &= \text{ext}_{A_p}^{k(m)} \circ \text{ext}_{A_p}^{\mathcal{O}} \circ \text{con}_{A_p}^K j(A^K) = \text{ext}_{A_p}^{k(m)} \circ \text{con}_{A_p}^K j(A^K) \\ &= \text{ext}_{A(p)}^{k(m)} \circ \text{ext}_{A_p}^{k(p)} \circ \text{con}_{A_p}^K j(A^K) = \text{ext}_{A(p)}^{k(m)} j(A(p)) \\ &= j(A^{\mathcal{O}}(\mathfrak{m})). \end{aligned}$$

As in the proof of 3.5 we used that  $\text{ext}_{A(p)}^{k(m)} j(A(p)) = j(A^{\mathcal{O}}(\mathfrak{m}))$  since  $A(p)$  splits. Since  $A^{\mathcal{O}}$  is  $\mathcal{O}$ -torsion free, also  $\text{con}_{A_p}^K j(A^K) \leq A^{\mathcal{O}}$  is  $\mathcal{O}$ -torsion free and thus free

since  $\mathcal{O}$  is a discrete valuation ring. Moreover, by 3.3 we have

$$\mathrm{ext}_{A^{\mathcal{O}}}^K \circ \mathrm{con}_{A^{\mathcal{O}}}^K(j(A^K)) = j(A^K)$$

and therefore  $\dim_{\mathcal{O}} \mathrm{con}_{A^{\mathcal{O}}}^K(j(A^K)) = \dim_K j(A^K)$ . Hence,

$$\begin{aligned} \dim_K j(A^K) &= \dim_{\mathcal{O}} \mathrm{con}_{A^{\mathcal{O}}}^K j(A^K) = \dim_{k(\mathfrak{m})} \mathrm{ext}_{A^{\mathcal{O}}}^{k(\mathfrak{m})} \circ \mathrm{con}_{A^{\mathcal{O}}}^K j(A^K) \\ &= \dim_{k(\mathfrak{m})} j(A^{\mathcal{O}}(\mathfrak{m})) = \dim_{k(\mathfrak{p})} j(A(\mathfrak{p})). \end{aligned}$$

This shows that  $\mathfrak{p} \in \mathrm{JacDimGen}(A)$  and so  $\mathcal{P}\mathrm{Gen}(A)$  is a generic neighborhood. The remaining properties on  $\mathcal{P}$  are now proven by similar arguments as in 6.2. ■

We can finally proof the main result of this article.

**6.4 Theorem.** If  $R$  is noetherian, then the set  $\mathrm{DecGen}(A) \cap \mathrm{Spl}(A)$  is an open generic neighborhood in  $\mathrm{Spec}(R)$ . In particular,  $\mathrm{DecGen}(A)$  is dense in  $\mathrm{Spec}(R)$ . If  $A$  has split fibers, then  $\mathrm{DecGen}(A)$  is open.

*Proof.* According to 5.5 and 6.3 both  $\mathrm{Spl}(A)$  and  $\mathrm{JacDimGen}(A)$  are open generic neighborhoods. Hence, their intersection is an open generic neighborhood and this is by 3.7 equal to  $\mathrm{DecGen}(A) \cap \mathrm{Spl}(A)$ . ■

**6.5 Remark.** Following the philosophy of the article it would be natural to prove that the property  $\mathcal{P}$  with  $\mathcal{P}(A, \mathfrak{p}) = 1$  if and only if  $d_A^{\mathfrak{p}}$  exists and is trivial defines a generic property. This would show that  $\mathrm{DecGen}(A)$  is not only dense but also open. But this approach seems to be out of range yet since for example for property 4.2(b)(iii) we would have to conclude from the fact that both  $d_A^{\mathfrak{p}} : G_0(A^K) \rightarrow G_0(A(\mathfrak{p}))$  and  $d_{A_{\mathfrak{p}}}^{\mathfrak{q}} : G_0(A(\mathfrak{p})) \rightarrow G_0(A(\mathfrak{q}))$  exist and are trivial that  $d_A^{\mathfrak{q}} : G_0(A^K) \rightarrow G_0(A(\mathfrak{q}))$  exists and is trivial. Of course the composition  $d_{A_{\mathfrak{p}}}^{\mathfrak{q}} \circ d_A^{\mathfrak{p}}$  exists and is trivial but we do not know if this composition is equal to  $d_A^{\mathfrak{q}}$ ! This property is only known to hold in general when  $R/\mathfrak{p}$  is normal (see [9]). This detail is also what makes it so hard to work with decomposition morphisms directly.

In case  $A$  is symmetric and  $A^K$  is semisimple, it is known that the Schur elements behave well under specializations and allow to detect when exactly a specialization is semisimple. We can combine this with our results to give a *precise* and *explicit* description of the complement  $\mathrm{DecEx}(A)$  of  $\mathrm{DecGen}(A)$ , thus completing the results known so far.

**6.6 Proposition.** Suppose that  $R$  is noetherian, that  $A$  has split fibers and is symmetric, and that  $A^K$  is semisimple. Let  $(c_i)_{i \in I}$  be the Schur elements of a system  $(S_i)_{i \in I}$  of representatives of the simple  $A^K$ -modules (see [8, §7]). Then

$$\mathrm{DecEx}(A) = \bigcup_{i \in I} V(c_i) = V\left(\prod_{i \in I} c_i\right).$$

*Proof.* As  $R$  is normal and  $A$  is symmetric, the Schur elements are contained in  $R$  by [8, 7.3.9] and so  $V(c_i)$  is well-defined. Suppose that  $\mathfrak{p}$  is not contained in any of the  $V(c_i)$ . Then the images of the  $c_i$  in  $A(\mathfrak{p})$  are all non-zero and so it follows from



Tits's deformation theorem 3.2 that  $A(\mathfrak{p})$  is semisimple and that  $d_A^{\mathfrak{p}}$  is trivial. Hence,  $\mathfrak{p} \in \text{DecGen}(A)$ . This shows that

$$\text{DecGen}(A) \supseteq \text{Spec}(R) \setminus \bigcup_{i \in I} V(c_i).$$

Now, let  $\mathfrak{p} \in \bigcup_{i \in I} V(c_i)$ . Then one Schur element of  $A(\mathfrak{p})$  is equal to zero and therefore  $A(\mathfrak{p})$  is not semisimple by [8, 7.2.6]. Hence,  $j(A(\mathfrak{p})) \neq 0$  and therefore  $\mathfrak{p} \notin \text{JacDimGen}(A) = \text{DecGen}(A)$  by 6.4. This shows the asserted equality. ■

**6.7 Question.** We raise several questions which should be investigated in the future as they have a considerable impact:

- (a) Is  $\text{DecGen}(A)$  open without assuming that  $A(\mathfrak{p})$  splits for all  $\mathfrak{p}$ ?
- (b) Is  $\text{DecGen}(A)$  dense without assuming that  $R$  is noetherian?
- (c) In case  $A(\mathfrak{p})$  splits we now know that  $\text{DecGen}(A)$  is open so that the complement  $\text{DecEx}(A)$  is closed and can thus be considered as a closed subscheme of  $\text{Spec}(R)$ . What geometric properties does  $\text{DecEx}(A)$  have? Is it pure of codimension one? Is there an explicit description of  $\text{DecEx}(A)$  as in 6.6? It looks like we have to find a proper generalization of Schur elements to non-semisimple algebras to answer this.

## References

- [1] Cédric Bonnafé and Raphaël Rouquier. *Cellules de Calogero–Moser*. 2013. URL: <http://arxiv.org/abs/1302.2720>.
- [2] Nicolas Bourbaki. *Commutative Algebra. Chapters 1 to 7*. Elements of mathematics. Translated from the French edition. Addison-Wesley Publishing Co., 1972.
- [3] Michel Broué, Gunter Malle, and Raphaël Rouquier. *Complex reflection groups, braid groups, Hecke algebras*. In: *J. Reine Angew. Math.* 500 (1998), pp. 127–190. DOI: 10.1515/crll.1998.064. URL: <http://www.degruyter.com/view/j/crll.1998.1998.issue-500/crll.1998.064/crll.1998.064.xml>.
- [4] Charles W. Curtis and Irving Reiner. *Methods of representation theory*. Vol. 1. New York: John Wiley & Sons, 1981.
- [5] Jie Du, Brian Parshall, and Leonard Scott. *Stratifying endomorphism algebras associated to Hecke algebras*. In: *Journal of Algebra* 203.1 (1998), pp. 169–210. DOI: 10.1006/jabr.1997.7325. URL: <http://dx.doi.org/10.1006/jabr.1997.7325>.
- [6] Pavel Etingof and Victor Ginzburg. *Symplectic reflection algebras, Calogero–Moser space, and deformed Harish–Chandra homomorphism*. In: *Invent. Math.* 147.2 (2002), pp. 243–348. URL: <http://arxiv.org/abs/math/0011114>.
- [7] Meinolf Geck. *Representations of Hecke algebras at roots of unity*. In: *Astérisque* 252 (1998). Séminaire Bourbaki. Vol. 1997/98, Exp. No. 836, 3, 33–55.
- [8] Meinolf Geck and Götz Pfeiffer. *Characters of finite Coxeter groups and Iwahori–Hecke algebras*. Vol. 21. London Mathematical Society Monographs. New Series. Oxford University Press, 2000.
- [9] Meinolf Geck and Raphaël Rouquier. *Centers and simple modules for Iwahori–Hecke algebras*. In: *Finite reductive groups (Luminy, 1994)*. Vol. 141. Progress in Mathematics. Birkhäuser Boston, 1997, pp. 251–272.
- [10] David M. Goldschmidt. *Lectures on character theory*. Publish or Perish Inc., 1980.
- [11] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Schemes with examples and exercises. Vieweg + Teubner, Wiesbaden, 2010.

- [12] Alexander Grothendieck. *Éléments de géométrie algébrique: II. Étude globale élémentaire de quelques classes de morphismes*. In: *Publications mathématiques de l'I.H.E.S.* 8 (1961), pp. 5–222. URL: [http://www.numdam.org/numdam-bin/feuilleter?id=PMIHES\\_1961\\_\\_8\\_](http://www.numdam.org/numdam-bin/feuilleter?id=PMIHES_1961__8_).
- [13] Alexander Grothendieck and Jean A. Dieudonné. *Éléments de géométrie algébrique. I*. Vol. 166. Die Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1971.
- [14] Tsit Yuen Lam. *A first course in noncommutative rings*. Vol. 131. Graduate Texts in Mathematics. New York: Springer-Verlag, 1991.
- [15] Tsit Yuen Lam. *Exercises in classical ring theory*. Second edition. Problem Books in Mathematics. Springer-Verlag, 2003.
- [16] Ulrich Thiel. *On restricted rational Cherednik algebras*. Dissertation, TU Kaiserslautern. 2014.